

# Stokes' Theorem

## Integral calculus

### Summary

**Line integrals.** For a vector field  $\boldsymbol{v}$  defined for all points of a curve  $\mathcal{C}$ ,  $\int_{\mathcal{C}} \boldsymbol{v} \cdot d\boldsymbol{l}$  is defined as

$$\int_{\mathcal{C}} \boldsymbol{v} \cdot d\boldsymbol{l} = \int_a^b (\boldsymbol{v}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)) dt ,$$

where  $\boldsymbol{\gamma}: [a, b] \rightarrow \mathcal{C}$  is any parameterization of  $\mathcal{C}$ .  $\int_{\mathcal{C}} \boldsymbol{v} \cdot d\boldsymbol{l}$  is also denoted as  $\tau_{\mathcal{C}}(\boldsymbol{v})$ . By its analogy with the work done by a force along a curve,  $\tau_{\mathcal{C}}(\boldsymbol{v})$  can be called the *toil* of  $\boldsymbol{v}$  along  $\mathcal{C}$ . In the case when  $\mathcal{C}$  is closed,  $\tau_{\mathcal{C}}(\boldsymbol{v}) = \int_{\mathcal{C}} \boldsymbol{v} \cdot d\boldsymbol{l}$  is also called the *circulation* of  $\boldsymbol{v}$  on  $\mathcal{C}$  (and in this case the integral is often written as  $\oint_{\mathcal{C}} \boldsymbol{v} \cdot d\boldsymbol{l}$ ).

**Fundamental Theorem of Calculus.** For any function  $f$ ,

$$\tau_{\mathcal{C}}(\partial f) = f(B) - f(A) ,$$

Where the points  $A$  and  $B$  are the origin and the end of  $\mathcal{C}$ , respectively.

**Surface integrals.** If  $\mathbf{v}$  is a vector field defined for all points of a surface  $\Sigma$ , the integral over  $\Sigma$  of  $\mathbf{v} \cdot d\boldsymbol{\sigma}$  is called the *flux* of  $\mathbf{v}$  through  $\Sigma$ , and is denoted  $\phi_{\Sigma}(\mathbf{v})$ :

$$\phi_{\Sigma}(\mathbf{v}) = \int_{\Sigma} \mathbf{v} \cdot d\boldsymbol{\sigma} .$$

The main result is Stokes' classical theorem: if  $\Sigma$  is an oriented surface with boundary  $\mathcal{C} = \partial\Sigma$ , and  $\mathbf{v}$  is a vector field defined on all points of  $\Sigma$ , then

$$\phi_{\Sigma}(\mathbf{curl}(\mathbf{v})) = \tau_{\mathcal{C}}(\mathbf{v}) .$$

For a parameterized surface,  $\boldsymbol{\sigma}: R \rightarrow \Sigma$ ,  $R$  a regular region in a plane, the flux can be computed with the following formula:

$$\phi_{\Sigma}(\mathbf{v}) = \int_R \mathbf{v}(\boldsymbol{\sigma}(s, t)) \cdot (\partial_s \boldsymbol{\sigma}(s, t) \times \partial_t \boldsymbol{\sigma}(s, t)) ds dt .$$

## ***Volume integrals*** $\int_W f(x, y, z) d\omega$

The main result is Gauss' divergence theorem: if  $W$  is a regular region in 3-space, possibly with boundary  $\partial W$ , and  $\boldsymbol{v}$  is a vector field defined on  $W$ , then

$$\int_W \operatorname{div}(\boldsymbol{v}) = \phi_{\partial W}(\boldsymbol{v}).$$