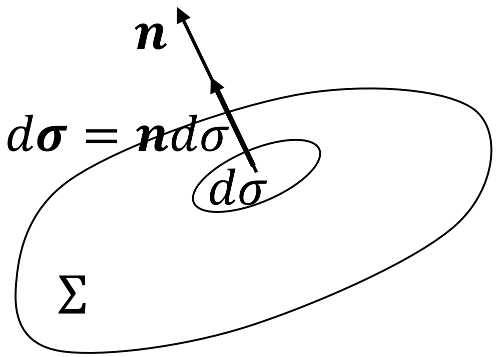
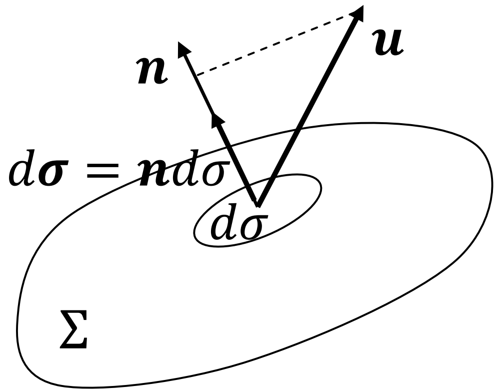
Stokes' Theorem

Integral calculus

Surface integrals



Let Σ be a surface, possibly with boundary. We assume that Σ is *oriented* by a smooth unit vector field $\boldsymbol{n}$, defined for all points of Σ, and orthogonal to Σ (we say that $\boldsymbol{n}$ is a *unit normal vector field* of Σ). If $d\sigma$ is an infinitesimal piece of Σ (usually called *area element*), then $d\boldsymbol{\sigma} = \boldsymbol{n}d\sigma$ is called the *vector area element* of Σ.



Flux

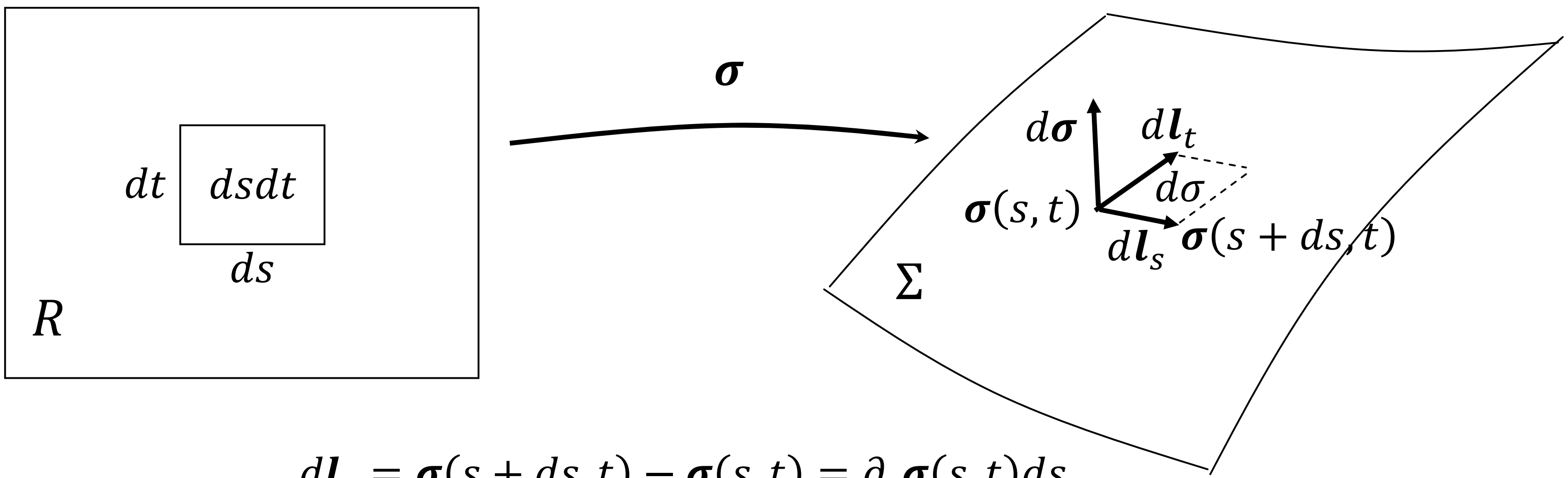
If $\boldsymbol{u}$ is a vector field defined for all points of Σ, the integral over Σ of $\boldsymbol{u} \cdot d\boldsymbol{\sigma}$ is called the *flux* of $\boldsymbol{u}$ trough Σ, and is denoted $\phi_\Sigma(\boldsymbol{u})$:

$$\phi_\Sigma(\boldsymbol{u}) = \int_\Sigma \boldsymbol{u} \cdot d\boldsymbol{\sigma} \, .$$

Example. For a parameterized surface, $\boldsymbol{\sigma}\colon R \to \Sigma$, R a regular region in a plane (say a rectangle with sides parallel to the coordinate axis),

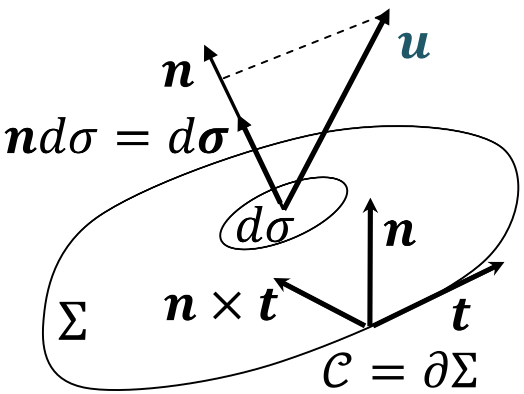
$$\phi_\Sigma(\boldsymbol{u}) = \int_R \boldsymbol{u}\big(\boldsymbol{\sigma}(s,t)\big) \cdot (\partial_s \boldsymbol{\sigma}(s,t) \times \partial_t \boldsymbol{\sigma}(s,t))dsdt.$$

Notice that $d\boldsymbol{l}_s = \partial_s \boldsymbol{\sigma}(s,t)ds$ is the vector arc element of the curve $s \mapsto \boldsymbol{\sigma}(s,t)$, t fixed, corresponding to ds (see the illustration on next page). Similarly, $d\boldsymbol{l}_t = \partial_t \boldsymbol{\sigma}(s,t)dt$ is the vector arc element of the curve $t \mapsto \boldsymbol{\sigma}(s,t)$, s fixed, corresponding to dt. Since the vector area spanned by $d\boldsymbol{l}_s$ and $d\boldsymbol{l}_t$ is given by the cross product $d\boldsymbol{l}_s \times d\boldsymbol{l}_t$, from the expressions of these vector arc elements we see that the vector area element corresponding to $dsdt$ is indeed $(\partial_s \boldsymbol{\sigma}(s,t) \times \partial_t \boldsymbol{\sigma}(s,t))dsdt$. So the integrand expression $\boldsymbol{u}\big(\boldsymbol{\sigma}(s,t)\big) \cdot (\partial_s \boldsymbol{\sigma}(s,t) \times \partial_t \boldsymbol{\sigma}(s,t))dsdt$ is the flux of $\boldsymbol{u}$ through that area element and the integral, which by definition is the sum of these elementary fluxes, yields the total flux of $\boldsymbol{u}$ through Σ.



$$d\boldsymbol{l}_s = \boldsymbol{\sigma}(s+ds,t) - \boldsymbol{\sigma}(s,t) = \partial_s\boldsymbol{\sigma}(s,t)ds$$

Example (Classical Stokes' theorem). Let Σ be a surface oriented by the unit normal vector $\boldsymbol{n}$ and with boundary $\mathcal{C} = \partial\Sigma$. Let $\boldsymbol{u}$ be a vector field defined on an open set containing Σ. Then



$$\phi_\Sigma\big(\mathbf{curl}(\boldsymbol{u})\big) = \tau_{\mathcal{C}}(\boldsymbol{u})\ ,$$

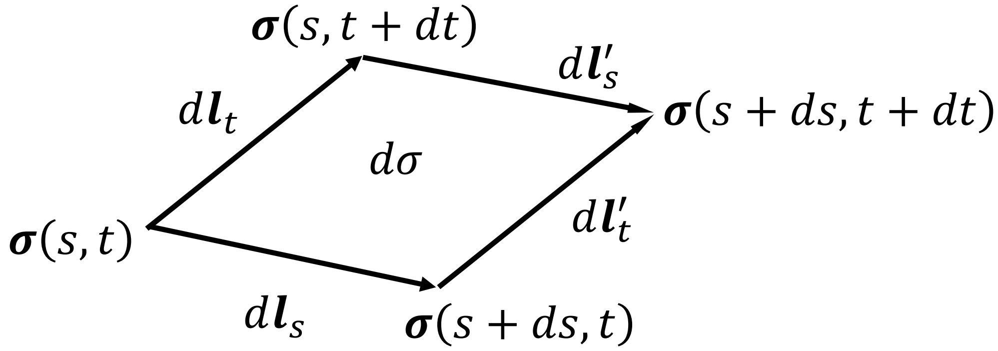
where $\mathcal{C}$ is oriented by the unit tangent vector $\boldsymbol{t}$ such that $\boldsymbol{n}\times\boldsymbol{t}$ is directed toward the interior of Σ.

Remark. Let us consider the ideas that classically were advanced as a proof of Stokes' theorem. This 'proof', which is still the one found in most Physics books, can be made quite rigorous, as we will see later on.

The basic reason why Stokes' theorem is true is that it holds for the area element $d\boldsymbol{\sigma}$ of the surface Σ. If this infinitesimal version is true, then adding up for all surface elements we get on one hand the flux of $\mathbf{curl}(\boldsymbol{u})$ through the surface and on the other the circulation of $\boldsymbol{u}$ around the boundary, as the net circulation when adding the circulations around the boundaries of the area elements (see below) is simply the circulation around the boundary of Σ.

Since to prove Stokes's theorem for $d\boldsymbol{\sigma}$ is a local question, we may assume that Σ is a parameterized surface as in the previous example, and hence, using the notations there, $d\boldsymbol{\sigma} = (\partial_s\boldsymbol{\sigma}(s,t) \times \partial_t\boldsymbol{\sigma}(s,t))dsdt$. For this surface element, the flux of $\mathbf{curl}(\boldsymbol{u})$ is simply

$$\mathbf{curl}(\boldsymbol{u}) \cdot d\boldsymbol{\sigma} = \mathbf{curl}(\boldsymbol{u}) \cdot (\partial_s\boldsymbol{\sigma}(s,t) \times \partial_t\boldsymbol{\sigma}(s,t))dsdt\ .$$



We want to see that this coincides with the circulation of $\boldsymbol{u}$ around the boundary of the area element $d\boldsymbol{\sigma}$ (see the illustration for a magnified version).

$$d\boldsymbol{l}_s = \boldsymbol{\sigma}(s+ds,t) - \boldsymbol{\sigma}(s,t) = \partial_s\boldsymbol{\sigma}(s,t)ds$$

The contribution of $d\boldsymbol{l}_s$ to this circulation is

$$\boldsymbol{u}\big(\boldsymbol{\sigma}(s,t)\big)\cdot d\boldsymbol{l}_s = \boldsymbol{u}\big(\boldsymbol{\sigma}(s,t)\big)\cdot \partial_s\boldsymbol{\sigma}(s,t)ds = \boldsymbol{u}\cdot\boldsymbol{\sigma}_s\, ds,$$

where to simplify notations we write $\boldsymbol{u}$ instead of $\boldsymbol{u}\big(\boldsymbol{\sigma}(s,t)\big)$ and $\boldsymbol{\sigma}_s$ instead of $\partial_s\boldsymbol{\sigma}(s,t)$. Similarly, the contribution of $d\boldsymbol{l}_t$ is $-\boldsymbol{u}\cdot\boldsymbol{\sigma}_t\, dt$.

Since $\boldsymbol{u}\big(\boldsymbol{\sigma}(s+ds,t)\big) = \boldsymbol{u}(\boldsymbol{\sigma}(s,t)+\boldsymbol{\sigma}_s(s,t)ds) = \boldsymbol{u} + (\boldsymbol{\partial u}\cdot\boldsymbol{\sigma}_s\,)ds$, the contribution of

$$d\boldsymbol{l}'_t = \boldsymbol{\sigma}(s+ds,t+dt) - \boldsymbol{\sigma}(s+ds,t) = \boldsymbol{\sigma}_t(s+ds,t)dt$$

$$= \boldsymbol{\sigma}_t dt + \boldsymbol{\sigma}_{ts} ds dt$$

to the circulation is $\boldsymbol{u}\cdot\boldsymbol{\sigma}_t\,dt+\boldsymbol{u}\cdot\boldsymbol{\sigma}_{ts}dsdt+(\boldsymbol{\partial u}\cdot\boldsymbol{\sigma}_s\,)\cdot\boldsymbol{\sigma}_t\,ds\,dt$ (we disregard the third order term). Similarly, the contribution of $d\boldsymbol{l}'_s$ to the circulation is $-\boldsymbol{u}\cdot\boldsymbol{\sigma}_s\,ds-\boldsymbol{u}\cdot\boldsymbol{\sigma}_{st}dsdt-(\boldsymbol{\partial u}\cdot\boldsymbol{\sigma}_t\,)\cdot\boldsymbol{\sigma}_s\,ds\,dt$. Adding up these four contributions we get, after cancelling terms, the following exact expression for the circulation:

$$(\boldsymbol{\partial u}\cdot\boldsymbol{\sigma}_s\,)\cdot\boldsymbol{\sigma}_t\,ds\,dt-(\boldsymbol{\partial u}\cdot\boldsymbol{\sigma}_t\,)\cdot\boldsymbol{\sigma}_s\,ds\,dt$$

(the third order terms vanish differentially in front of the second order terms). Finally it is straightforward to check that

$$(\boldsymbol{\partial u}\cdot\boldsymbol{\sigma}_s\,)\cdot\boldsymbol{\sigma}_t-(\boldsymbol{\partial u}\cdot\boldsymbol{\sigma}_t\,)\cdot\boldsymbol{\sigma}_s$$

agrees with $\mathbf{curl}(\boldsymbol{u})\cdot\big(\partial_s\boldsymbol{\sigma}(s,t)\times\partial_t\boldsymbol{\sigma}(s,t)\big)=\mathbf{curl}(\boldsymbol{u})\cdot(\boldsymbol{\sigma}_s\times\boldsymbol{\sigma}_t)$. The calculation can be done as follows:

$$\begin{aligned}(\boldsymbol{\partial u}\cdot\boldsymbol{\sigma}_s\,)\cdot\boldsymbol{\sigma}_t &= \big(\boldsymbol{\partial}u_x\cdot\boldsymbol{\sigma}_s,\boldsymbol{\partial}u_y\cdot\boldsymbol{\sigma}_s,\boldsymbol{\partial}u_z\cdot\boldsymbol{\sigma}_s\big)\cdot\boldsymbol{\sigma}_t\\
&= \big(\partial_x u_x\sigma_{xs}+\ \partial_y u_x\sigma_{ys}+\partial_z u_x\sigma_{zs}\big)\sigma_{xt}\\
&\quad+\big(\partial_x u_y\sigma_{xs}+\ \partial_y u_y\sigma_{ys}+\partial_z u_y\sigma_{zs}\big)\sigma_{yt}\\
&\quad+\big(\partial_x u_z\sigma_{xs}+\ \partial_y u_z\sigma_{ys}+\partial_z u_z\sigma_{zs}\big)\sigma_{zt}.\end{aligned}$$

Similarly, $(\boldsymbol{\partial u} \cdot \boldsymbol{\sigma}_t\,) \cdot \boldsymbol{\sigma}_s =$

$$\left(\partial_x u_x \sigma_{xt} + \partial_y u_x \sigma_{yt} + \partial_z u_x \sigma_{zt}\right)\sigma_{xs}$$

$$+\left(\partial_x u_y \sigma_{xt} + \partial_y u_y \sigma_{yt} + \partial_z u_y \sigma_{zt}\right)\sigma_{ys}$$

$$+\left(\partial_x u_z \sigma_{xt} + \partial_y u_z \sigma_{yt} + \partial_z u_z \sigma_{zt}\right)\sigma_{zs}.$$

On subtracting the two expressions we see that $(\boldsymbol{\partial u} \cdot \boldsymbol{\sigma}_s\,) \cdot \boldsymbol{\sigma}_t - (\boldsymbol{\partial u} \cdot \boldsymbol{\sigma}_t\,) \cdot \boldsymbol{\sigma}_s$ is equal to

$$\partial_y u_z \sigma_{ys} \sigma_{zt} - \partial_z u_y \sigma_{zt} \sigma_{ys} - \partial_y u_z \sigma_{yt} \sigma_{zs} + \partial_z u_y \sigma_{zs} \sigma_{yt} + \cdots$$

$$\left(\partial_y u_z - \partial_z u_y\right)\left(\sigma_{ys}\sigma_{zt} - \sigma_{zs}\sigma_{yt}\right) + \cdots \ ,$$

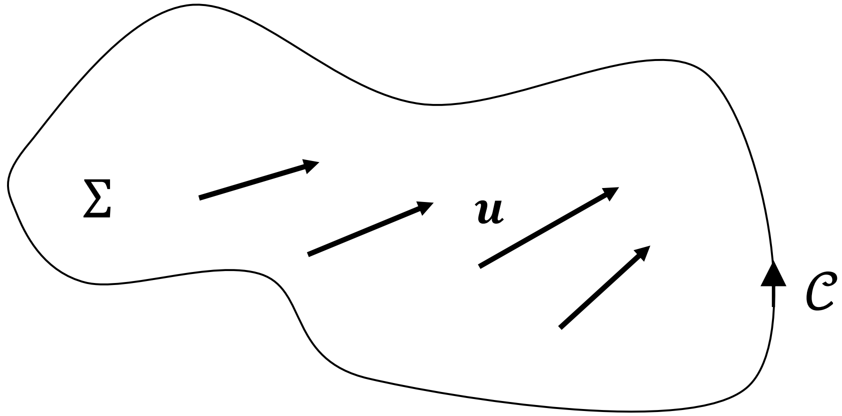
where $\cdots$ denote the terms obtained from the written terms by cyclic permutation of x, y, z.

But $\partial_y u_z - \partial_z u_y = \mathbf{curl}(\boldsymbol{u})_x$, $\sigma_{ys}\sigma_{zt} - \sigma_{zs}\sigma_{yt} = (\boldsymbol{\sigma}_s \times \boldsymbol{\sigma}_t)_x$, and therefore

$$(\boldsymbol{\partial v} \cdot \boldsymbol{\sigma}_s\,) \cdot \boldsymbol{\sigma}_t - (\boldsymbol{\partial v} \cdot \boldsymbol{\sigma}_t\,) \cdot \boldsymbol{\sigma}_s$$

$$= \mathbf{curl}(\boldsymbol{v})_x (\boldsymbol{\sigma}_s \times \boldsymbol{\sigma}_t)_x + \cdots$$

$= \mathbf{curl}(\boldsymbol{v}) \cdot (\boldsymbol{\sigma}_s \times \boldsymbol{\sigma}_t)$, as claimed.



Green's theorem

Let us consider the special case of Stokes' theorem in which $\mathcal{C}$ is a simple closed curve in the plane xy, Σ is the region in that plane enclosed by $\mathcal{C}$ and $\boldsymbol{u} = (P(x,y), Q(x,y), 0)$, where $P(x,y)$ and $Q(x,y)$ are differentiable functions. Then

$$\mathbf{curl}(\boldsymbol{u}) = \left(0,0,\partial_x Q - \partial_y P\right)$$

and $\phi_\Sigma\left(\mathbf{curl}(\boldsymbol{u})\right) = \int_\Sigma \left(\partial_x Q - \partial_y P\right) dxdy$ (orienting Σ by $\boldsymbol{e}_z = (1,0,0)$). On the other hand we see that

$$[*] \qquad \tau_{\mathcal{C}}(\boldsymbol{u}) = \int_a^b \Big(P\big(x(t), y(t)\big)\gamma_x'(t) + Q\big(x(t), y(t)\big)\gamma_y'(t)\Big)\, dt,$$

where $\boldsymbol{\gamma}(t) = \left(\gamma_x(t), \gamma_y(t)\right)$, $a \leq t \leq b$, is a parameterization of $\mathcal{C}$. This last integral can be denoted $\int_{\mathcal{C}} Pdx + Qdy$, as this expression allows us to form [$*$] as soon as we have the parameterization $\left(x(t), y(t)\right)$ of $\mathcal{C}$ by using the rules $d\left(\gamma_x(t)\right) = \gamma_x'(t)dt$ and $d\left(\gamma_y(t)\right) = \gamma_y'(t)dt$.

In conclusion, we see that

$$\int_{\mathcal{C}} Pdx + Qdy = \int_{\Sigma} \left(\partial_x Q - \partial_y P\right) dxdy,$$

which is usually called *Green's theorem*.