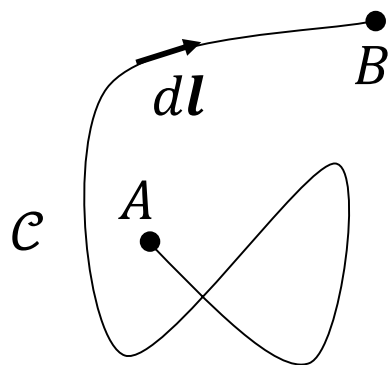


Stokes' Theorem

Integral calculus

Line integrals



Let \mathcal{C} be a space curve joining the points A and B (the possibility $B = A$ is allowed). Take any parameterization $\gamma: [a, b] \rightarrow \mathcal{C}$, which means that when t runs over the interval $[a, b] \subseteq \mathbb{R}$, $\gamma(t)$ runs over all the points of \mathcal{C} , with $\gamma(a) = A$ and $\gamma(b) = B$ (γ is supposed to be continuous

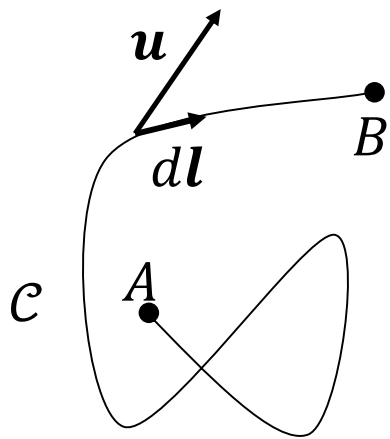
and piecewise differentiable, but need not be one-to-one). For an infinitesimal increment dt of t , we have

$$\gamma(t + dt) = \gamma(t) + \gamma'(t)dt .$$

This means that the vector

$$dl = \gamma'(t)dt = \gamma(t + dt) - \gamma(t)$$

represents the infinitesimal element of \mathcal{C} from $\boldsymbol{\gamma}(t)$ to $\boldsymbol{\gamma}(t + dt)$, and it is called the (vector) *line element* of \mathcal{C} (the *scalar line element*, dl , is the norm of $d\mathbf{l}$, namely $dl = |\boldsymbol{\gamma}'(t)|dt$).



Let now \mathbf{u} be a vector field defined for all points of \mathcal{C} . If \mathbf{u} were a force, the scalar product $\mathbf{u} \cdot d\mathbf{l}$ would represent the work done by \mathbf{u} over the (infinitesimal) displacement $d\mathbf{l}$, and the total work done by \mathbf{u} along \mathcal{C} would be

$$\int_a^b \mathbf{u} \cdot d\mathbf{l} = \int_a^b (\mathbf{u} \cdot \boldsymbol{\gamma}'(t)) dt .$$

This integral makes sense for any \mathbf{u} , depends only of \mathcal{C} and \mathbf{u} , and is denoted $\int_{\mathcal{C}} \mathbf{u} \cdot d\mathbf{l}$, or also $\tau_{\mathcal{C}}(\mathbf{u})$, and it may be called the *toil of \mathbf{u}* along \mathcal{C} . In other words,

$$\tau_{\mathcal{C}}(\mathbf{u}) = \int_{\mathcal{C}} \mathbf{u} \cdot d\mathbf{l} = \int_a^b (\mathbf{u}(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t)) dt ,$$

where $\boldsymbol{\gamma}$ is any parameterization of \mathcal{C} .

If we want to stress the parameterization, we will write $\tau_{\boldsymbol{\gamma}}(\mathbf{u})$ and $\int_{\boldsymbol{\gamma}} \mathbf{u} \cdot d\mathbf{l}$.

Example (The Fundamental Theorem of Calculus, FTC). For any function f ,

$$\tau_{\mathcal{C}}(\boldsymbol{\partial}f) = f(B) - f(A).$$

In particular we see that $\tau_{\mathcal{C}}(\boldsymbol{\partial}f)$ only depends on the end points A and B of \mathcal{C} , and not on the particular curve \mathcal{C} used to go from A to B .

Proof. If $\boldsymbol{\gamma}: [a, b] \rightarrow \mathcal{C}$ is a parameterization of \mathcal{C} , and $\gamma_x, \gamma_y, \gamma_z$ are the components of $\boldsymbol{\gamma}$, then

$$\boldsymbol{\partial}f \cdot d\mathbf{l} = \boldsymbol{\partial}f \cdot \boldsymbol{\gamma}'(t)dt = (\partial_x f)\gamma'_x(t)dt + (\partial_y f)\gamma'_y(t)dt + (\partial_z f)\gamma'_z(t)dt.$$

But the last expression is equal to $h'(t)dt$, with $h(t) = f(\boldsymbol{\gamma}(t))$, as direct consequence of the chain rule. Therefore we have

$$\tau_{\mathcal{C}}(\boldsymbol{\partial}f) = \tau_{\boldsymbol{\gamma}}(\boldsymbol{\partial}f) = \int_a^b \boldsymbol{\partial}f \cdot d\mathbf{l} = \int_a^b \boldsymbol{\partial}f \cdot \boldsymbol{\gamma}'(t) dt = \int_a^b h'(t)dt.$$

But by the usual FTC for functions in one variable,

$$\int_a^b h'(t)dt = h(b) - h(a) = f(\boldsymbol{\gamma}(b)) - f(\boldsymbol{\gamma}(a)) = f(B) - f(A).$$

Remark. The usual FTC is a special case of the FTC above.

Indeed, let $f = f(x)$ is a real valued function of one variable

$$x \in [a, b] \subseteq \mathbb{R}.$$

If we consider f , as we may, as a function of x, y, z , then

$$\mathbf{d}f = (f'(x), 0, 0).$$

On the other hand we may consider the interval $[a, b]$ as a curve with end points $A = (a, 0, 0)$ and $B = (b, 0, 0)$, so that in particular

$$f(B) - f(A) = f(b) - f(a).$$

If we parameterize this curve by $\boldsymbol{\gamma}(t) = (t, 0, 0)$, $t \in [a, b]$, then $\boldsymbol{\gamma}'(t) = (1, 0, 0) = \mathbf{e}_x$, $d\mathbf{l} = \mathbf{e}_x dt$ and $\int_a^b \mathbf{d}f \cdot d\mathbf{l} = \int_a^b f'(t) dt$.

Thus we see that the FTC in the form $\int_a^b \mathbf{d}f \cdot d\mathbf{l} = f(B) - f(A)$ yields

$$\int_a^b f'(t) dt = f(b) - f(a)$$

in our special case.