Stokes' Theorem

Notations and Conventions

Divergence

The *divergence* of a vector field $\boldsymbol{v} = \left(v_x, v_y, v_z\right)$ is the function

$$\operatorname{div}(\boldsymbol{v}) = \partial_x v_x + \partial_y v_y + \partial_z v_z \,.$$

Symbolically,

$$\operatorname{div}(\boldsymbol{v}) = \boldsymbol{\partial} \cdot \boldsymbol{v}$$

(or $\boldsymbol{\nabla} \cdot \boldsymbol{v}$ in the "nabla" notation).

Example. $\operatorname{div}\left(\mathbf{curl}(\boldsymbol{v})\right) = 0$ (The divergence of a curl is 0).

Proof. If $\boldsymbol{w} = \mathbf{curl}(\boldsymbol{v}) = (\partial_y v_z - \partial_z v_y, \partial_z v_x - \partial_x v_z, \partial_x v_y - \partial_y v_x)$, then

$$\begin{aligned}\operatorname{div}(\boldsymbol{w}) &= \partial_x\left(\partial_y v_z - \partial_z v_y\right) + \partial_y(\partial_z v_x - \partial_x v_z) + \partial_z\left(\partial_x v_y - \partial_y v_x\right) \\ &= \partial_x\left(\partial_y v_z\right) - \partial_y(\partial_x v_z) + \cdots = 0\end{aligned}$$

(by Schwarz's theorem).

Exemple. Let f be a function. Then

$$\operatorname{div}(\boldsymbol{\partial} f) = \partial_x(\partial_x f) + \partial_y\left(\partial_y f\right) + \partial_z(\partial_z f) = \partial_x^2 f + \partial_y^2 f + \partial_z^2 f.$$

This is usually written symbolically as

$$\operatorname{div}(\boldsymbol{\partial} f) = \Delta f,$$

with $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$, and Δ is called the *laplacian* operator. It is also written $\boldsymbol{\partial}^2$ (or $\boldsymbol{\nabla}^2$ in the "nabla" notation).