Stokes' Theorem

Notations and Conventions

Curl

The *curl* of a vector field $\boldsymbol{v} = \left(v_x, v_y, v_z\right)$ is the vector field

$$\mathbf{curl}(\boldsymbol{v}) = \left(\partial_y v_z - \partial_z v_y, \partial_z v_x - \partial_x v_z, \partial_x v_y - \partial_y v_x\right).$$

Symbolically,

$$\mathbf{curl}(\boldsymbol{v}) = \boldsymbol{\partial} \times \boldsymbol{v}$$

(or $\boldsymbol{\nabla} \times \boldsymbol{v}$ in the "nabla" notation).

Example. If $\boldsymbol{v}(x, y, z) = (yz, x^2z^2, xy^2z)$, then

$$\mathbf{curl}(f) = (2yxz - 2x^2z, y - y^2z, 2xz^2 - z).$$

Example. Let $\boldsymbol{u} = (a, b, c)$ be a vector (thus a, b, c are constants), $\boldsymbol{r} = (x, y, z)$, and $\boldsymbol{v} = \boldsymbol{u} \times \boldsymbol{r} = (bz - cy, cx - az, ay - bx)$. Then

$$\mathbf{curl}(\boldsymbol{v}) = 2\boldsymbol{u}\ .$$

Indeed,

$$\partial_y v_z = \partial_y(ay - bx) = a, \partial_z v_y = \partial_z(cx - az) = -a,$$

and so $\partial_y v_z - \partial_z v_y = 2a$. The relations

$$\partial_z v_x - \partial_x v_z = 2b \text{ and } \partial_x v_y - \partial_y v_x = 2c$$

are obtained in a similar way. Hence

$$\mathbf{curl}(\boldsymbol{v}) = (2a, 2b, 2c) = 2\boldsymbol{u}\ ,$$

as claimed.

Example. $\mathbf{curl}(\boldsymbol{\partial} f) = 0$ (The curl of a gradient is 0).

Proof. If $\boldsymbol{v} = \boldsymbol{\partial} f = (\partial_x f, \partial_y f, \partial_z f)$, then

$$\partial_y v_z - \partial_z v_y = \partial_y(\partial_z f) - \partial_z(\partial_y f) = 0$$

(by Schwarz's theorem, $\partial_y(\partial_z f) = \partial_z(\partial_y f)$).

That the y and z components of $\mathbf{curl}(\boldsymbol{\partial} f)$ are 0 is seen in a similar way.