Stokes' Theorem

Notations and Conventions

Gradient

The *gradient* of a function[1] $f = f(x, y, z)$ is the vector field[2]

$$\boldsymbol{\partial} f = \left(\partial_x f, \partial_y f, \partial_z f\right),$$

where $\partial_x f = \partial f/\partial x$ (the partial derivative of f with respect to x), and similarly with $\partial_y f$ and $\partial_z f$.

Other notations for $\boldsymbol{\partial} f$ are $\boldsymbol{\nabla} f$ and $\mathbf{grad}(f)$.

Example. If $f(x, y, z) = x^3 + y^2 z^2 + xyz$, then

$$\boldsymbol{\partial} f = (3x^2 + yz, 2yz^2 + xz, 2y^2 z + xy).$$

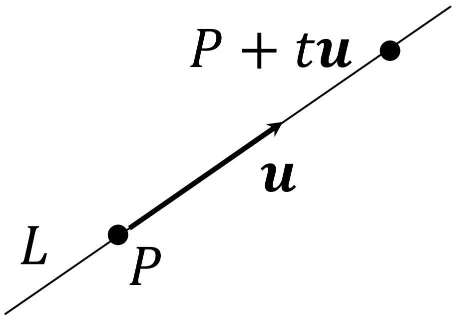
[1] In these notes, function means a differentiable function of its arguments.

[2] A vector field is a vector-valued function.

Interpretation

Fix a point $P(x, y, z)$ and a vector $\boldsymbol{u} = (a, b, c)$. Consider the function

$$h(t) = f(P + t\boldsymbol{u}) = f(x + at, y + bt, z + ct)\ .$$



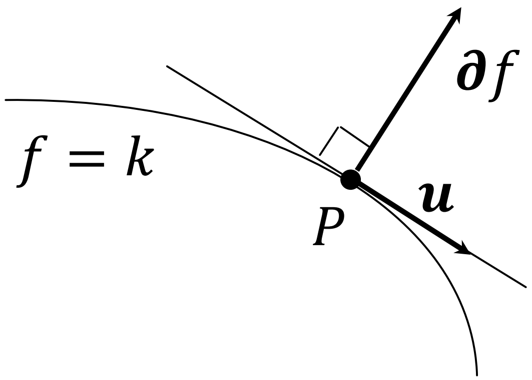
This function encodes the variation of f when restricted to the line L that goes through P with direction vector $\boldsymbol{u}$. Then the derivative of $h'(0)$ represents the *rate of change* of f at P in the direction $\boldsymbol{u}$. This rate is denoted $(D_{\boldsymbol{u}}f)(P)$, or just $D_{\boldsymbol{u}}f$ if P is clear from the context, and it is also called the *derivative of f at P in the direction $\boldsymbol{u}$.* But by the chain rule,

$$h'(0) = a\partial_x f + b\partial_y f + c\partial_z f = \boldsymbol{u} \cdot \boldsymbol{\partial} f$$

Therefore, *for any vector $\boldsymbol{u}$, the scalar product $\boldsymbol{u} \cdot \boldsymbol{\partial} f$ gives the rate of change of f in the direction $\boldsymbol{u}$* , or

$$\boldsymbol{u} \cdot \boldsymbol{\partial} f = D_{\boldsymbol{u}} f\,.$$

As a result, the rate of change of f is maximum, zero and minimum for vectors that are respectively parallel, perpendicular and antiparallel to the gradient.



In particular we see that $\boldsymbol{\partial} f$ is orthogonal to the level surfaces of f. Indeed, a *level surface* of f is a surface of the form $f = k$, where k is a constant, and the vectors tangent to this surface at P are the vectors $\boldsymbol{u}$ such that $D_{\boldsymbol{u}} f = 0$. Thus

$$\boldsymbol{u} \cdot \boldsymbol{\partial} f = D_{\boldsymbol{u}} f = 0$$

for any tangent vector $\boldsymbol{u}$ to the level surface at P.

Example. The level surface $x^2 + y^2 + z^2 = r^2$ (r a positive constant) of the function $f = x^2 + y^2 + z^2$ is the sphere of radius r with center at the origin. The outer normal unit vector of this sphere at $P = (x, y, z)$ is the vector $(x/r, y/r, z/r)$. On the other hand $\boldsymbol{\partial} f = (2x, 2y, 2z)$, confirming that it is perpendicular to the sphere.

Remark. If we set $\boldsymbol{u} = \boldsymbol{e}_x$ in the formula

$$\boldsymbol{u} \cdot \boldsymbol{\partial} f = D_{\boldsymbol{u}} f \ ,$$

where $\boldsymbol{e}_x = (1,0,0)$ is the unit vector in the x axis direction, we get

$$\partial_x f = D_{\boldsymbol{e}_x} f \ ,$$

in agreement with the fact that $\partial_x f = \partial f / \partial x$ is the rate of change of f in the direction of the x-axis. A similar comment is valid for the other two coordinate axes.